Admissibility
Projectivity and active structural completeness

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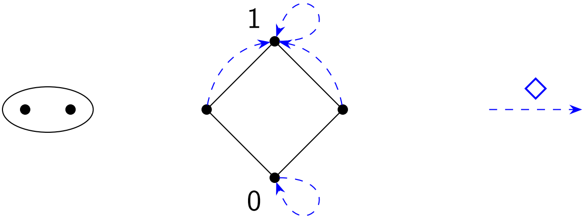
Let $\boldsymbol{S}_2$ be the depicted simple modal algebra. It is the dual of the two element cluster frame $\mathbb{C}_2$.



Let L be a modal logic such that $\mathbb{C}_2 \vDash L$ (equiv., $\boldsymbol{S}_2 \in \mathcal{V}_{\mathrm{L}}$) and $\Diamond 1 \in L$ (equiv., $\mathbf{2} \in V_{\mathrm{L}}$). Then the rule $\dfrac{\Diamond x, \Diamond \neg x}{0}$ is admissible but not derivable in L.

Exercise

Justify the statement.

Analogical argument shows that most of varieties of MV-algebras are not structurally complete.
Let $\boldsymbol{L}_n$ be the $n+1$-element chain MV-algebra:
$L_n = \{0, \ldots, n\}$ with operations $\neg a = n - a$, $a \oplus b = min(a + b, n)$, $a \wedge b = \min(a, b)$, and $1 = n$.
Then every variety $\mathcal{V}$ containing $\boldsymbol{L}_n$, $n > 1$, is not structurally complete. The quasi-identity

$$\forall x \; \overbrace{(x \wedge \neg x) \oplus \cdots \oplus (x \wedge \neg x)}^{n \text{ times}} = 1 \Rightarrow 1 = \neg 1$$

fails in $\boldsymbol{L}_n$, and hence is not derivable in $\mathcal{V}$. But its premise is not satisfiable in $\boldsymbol{L}_1 \leqslant \boldsymbol{L}_n$. Thus it is admissible.

These two examples are **odd** from the point of view of admissible rules.
Remember that admissibility appears in the context of proof systems for theoremhood of a deductive system. But the presented rules cannot be used in any proof of a theorem.

Definition (Wroński)

A rule Γ/δ is **passive** in a consequence relation $\vdash$ if there is no substitution s such that $\vdash s(\gamma)$ for all $\gamma \in \Gamma$.
A quasi-identity $\forall \bar{x} \bigwedge \varphi_i = \psi_i \Rightarrow \varphi = \psi$ is **passive** in a quasivariety $\mathcal{Q}$ if there is no substitution s such that $\mathcal{Q} \models \forall \bar{x} \bigwedge s(\psi_i) = s(\varphi_i)$.

Definition (Dzik)

A rule is **active** in $\vdash$ if it is not passive.
A quasi-identity is **active** in $\mathcal{Q}$ if it is not passive.

Definition (Dzik)

A consequence relation is **actively structurally complete** if every its admissible and active rule is derivable.
A quasivariety is **actively structurally complete** if every its admissible and active quasi-identity is derivable.

Finitely presented algebras

Definition

Let $\mathcal{Q}$ be a quasivariety. A **finitely presented algebra in $\mathcal{Q}$** is any algebra of the form $\boldsymbol{Fr}_{\mathcal{Q}}(X)/\theta$, where X is finite and θ is a finitely generated $\mathcal{Q}$-congruence.

Notation:
Recall that elements of $\boldsymbol{Fr}_{\mathcal{Q}}(X)$ may be represented by formulas.
We may abuse the notation and write
$\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/(\varphi_1 = \psi_1, \ldots, \varphi_k = \psi_k)$, or $\boldsymbol{Fr}_{\mathcal{Q}}/E$, for
$\boldsymbol{Fr}_{\mathcal{Q}}(X)/\theta_{\mathcal{Q}}((\varphi_1, \psi_1), \ldots, (\varphi_k, \psi_k))$, where $\varphi_i, \psi_i \in \boldsymbol{Form}(X)$ and $\bar{x}$ is a list of variables in X.
Sometimes we may also write $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/\varphi$ for $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/(\varphi = 1)$.

Exercise

Verify that every finite algebra (in a finite language) is finitely presented in any quasivariety.

Finitely presented algebra

Fact

Finitely presented algebras may be defined by the following universal property. An algebra $\boldsymbol{P} \in \mathcal{Q}$ if finitely presented by the defining equations $\varphi_1 = \psi_1, \ldots, \varphi_k = \psi_k$, with $\varphi_i, \psi_i \in \boldsymbol{Form}(X)$ for a finite X, if there exists a homomorphism $e\colon \boldsymbol{Form}(X) \to \boldsymbol{P}$ such that for every homomorphism $v\colon \boldsymbol{Form}(\bar{x}) \to \boldsymbol{A} \in \mathcal{Q}$ satisfying $v(\varphi_i) = v(\psi_i)$ for all i, there exists a unique homomorphism $\bar{v}\colon \boldsymbol{Fr}_{\mathcal{Q}}/(\varphi_1 = \psi_1, \ldots, \varphi_k = \psi_k) \to \boldsymbol{A}$ such that $v = \bar{v} \circ e$

$$\begin{tikzcd}
 & \boldsymbol{Form}(X) \arrow[dl, "e"'] \arrow[dr, "v"] & \\
\boldsymbol{P} \arrow[rr, dashed, "\exists ! \bar{v}"'] & & \boldsymbol{A}
\end{tikzcd}$$

As in the case of free algebras, we commonly omit mentioning e. Consequently, elements of $\boldsymbol{P}$ are represented by formulas.

Recall that $\mathcal{Q} \models \forall \bar{x}\; \gamma = \delta$ iff $\boldsymbol{Fr}_{\mathcal{Q}} \models \gamma = \delta$.
We have similar fact for quasi-identities.

Fact This is important!

Let $\forall \bar{x} \bigwedge \varphi_i = \psi_i \Longrightarrow \varphi = \psi$ be a quasi-identity, $\mathcal{Q}$ be a quasivariety, and $\boldsymbol{P} \in \mathcal{Q}$ be finitely presented in $\mathcal{Q}$ with the defining equations $\varphi_1 = \psi_1, \ldots, \varphi_k = \psi_k$. Then

$$\mathcal{Q} \models \forall \bar{x} \bigwedge \varphi_i = \psi_i \Longrightarrow \varphi = \psi \quad \text{iff} \quad \boldsymbol{P} \models \varphi = \psi.$$

Fact, reformulation

Let $\forall \bar{x} \bigwedge \varphi_i = \psi_i \Rightarrow \varphi = \psi$ be a quasi-identity, $\mathcal{Q}$ be a quasivariety. Then

- q is derivable in $\mathcal{Q}$ iff $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/(\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n) \models \varphi = \psi$;
- q is admissible in $\mathcal{Q}$ iff $\boldsymbol{Fr}_{Q(\boldsymbol{Fr}_Q)}(\bar{x})/(\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n) \models \varphi = \psi$;

Since $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/(\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n)$ is a homomorphic image of $\boldsymbol{Fr}_{Q(\boldsymbol{Fr}_Q)}(\bar{x})/(\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n)$, we obtain the following

Corollary

A quasivariety is structurally complete iff every its finitely presented algebra is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

Fact

Let $\mathcal{Q}$ be a quasivariety and $q : \forall \bar{x} \bigwedge \varphi_i = \psi_i \Rightarrow \varphi = \psi$,
$\boldsymbol{P}$ a finitely presented algebras with the defining equations form the premise of q. Then TFAE:

1. q is active in $\mathcal{Q}$;
2. There is a homomorphism $h\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{Q}}$;
3. There is a homomorphism $h\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{Q}}(0)$ (if there is a constant in the language);

Proof.

1⇒2 Let s be a substitution. Then
$\mathcal{Q} \models \forall \bar{x} \bigwedge s(\varphi_i) = s(\psi_i)$ iff $\boldsymbol{Fr}_{\mathcal{Q}} \models \bigwedge s(\varphi_i) = s(\psi_i)$, and

$$\begin{tikzcd} & \boldsymbol{Form}(X) \arrow[dl, "e"'] \arrow[dr, "s"] & \\ \boldsymbol{P} \arrow[rr, dashed, "h=\bar{s}"'] & & \boldsymbol{Fr}_{\mathcal{Q}} \end{tikzcd}$$

Active quasi-identities characterized

TFAE:

1. q is active in $\mathcal{Q}$;
2. There is a homomorphism $h\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{Q}}$;
3. There is a homomorphism $h\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{Q}}(0)$ (if there is a constant in the language);

Proof.

1⟹2 Let s be a substitution. Then

$\mathcal{Q} \models \forall \bar{x} \bigwedge s(\varphi_i) = s(\psi_i)$ iff $\boldsymbol{Fr}_{\mathcal{Q}} \models \bigwedge s(\varphi_i) = s(\psi_i)$, and

$$\begin{tikzcd} & \boldsymbol{Form}(X) \arrow[dl, "e"'] \arrow[dr, "s"] & \\ \boldsymbol{P} \arrow[rr, dashed, "h=\bar{s}"'] & & \boldsymbol{Fr}_{\mathcal{Q}} \end{tikzcd}$$

2⟹1 Define $s = h \circ e$.

2⟺3 Since there are homomorphisms $\boldsymbol{Fr}_{\mathcal{Q}} \to \boldsymbol{Fr}_{\mathcal{Q}}(0) \to \boldsymbol{Fr}_{\mathcal{Q}}$.

Corollary

A quasivariety is structurally complete iff every its finitely presented algebra, that admits a homomorphism into $\boldsymbol{Fr}_Q$, is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

Theorem (Bergman)

A quasivariety $\mathcal{Q}$ is structurally complete

iff every nontrivial $\mathcal{Q}$-finitely presented algebra is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

iff every $\mathcal{Q}$-subdirectly irreducible algebra is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

Theorem (Dzik & S., Metcalfe & Röthlisberger)

Let $\mathcal{Q}$ be a quasivariety and $\boldsymbol{C}$ be any subalgebra of $\boldsymbol{Fr}_{Q}$.
Then $\mathcal{Q}$ is actively structurally complete

iff for every $\mathcal{Q}$-finitely presented algebra, if $\boldsymbol{P}$ admits a homomorphism into $\boldsymbol{C}$ then $\boldsymbol{P} \in Q(\boldsymbol{Fr}_{\mathcal{Q}})$;

iff for every $\mathcal{Q}$-subdirectly irreducible algebra $\boldsymbol{S}$, $\boldsymbol{S} \times \boldsymbol{C} \in Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

Theorem (Dzik & S., Metcalfe & Röthlisberger)

Let $\mathcal{Q}$ be a quasivariety and $\boldsymbol{C}$ be any subalgebra of $\boldsymbol{Fr}_Q$.
Then $\mathcal{Q}$ is actively structurally complete iff
$\boldsymbol{S} \times \boldsymbol{C} \in \mathcal{Q}(\boldsymbol{Fr}_{\mathcal{Q}})$ for each $\mathcal{Q}$-subdirectly irreducible algebra $\boldsymbol{S}$.

A nice corollary

Let L be a locally tabular modal logic with a theorem $\Diamond 1 = 1$, this is, that is $\bullet$ is an L-frame. Then L is actively structurally complete iff, for every finite rooted element L-frame

$$\begin{array}{c}\text{either } \mathbb{F} \text{ is a p-morphic image of } \mathbb{U}(\log|F|) \\ \text{or } \mathbb{F} \uplus \bullet \text{ is a p-morphic image of } \mathbb{U}(\log|F|).\end{array}$$

Exercise

Check that the logic of two element cluster $\mathbb{C}_2$ is actively structurally complete. You may also want to check that S5 is actively structurally complete.

With the use of the previous corollary, together with ideas for Metcalfe, Röthlisberger, one can check (active) structural for completeness for small algebras and frames. The table below show a sample result of this sort.

size	logics	SC logics	ASC logics
1	1	100 %	100 %
2	5	40.0 %	100 %
3	62	24.2 %	85.5 %
4	2214	12.1 %	74.4 %
5	244134	7.9 %	76.4 %
6	87722854	5.7 %	83.3 %

Table: Normal extensions of $K \oplus \Diamond 1$ (S., Uliński)

active version of structural completeness was introduced decades later than the standard version?
Admissibility was studied mostly in the context of intuitionistic and intermediate logics.

Prime power theorem

Every nontrivial Heyting algebra has a prime filter. Equivalently, Every nontrivial Heyting algebra admits a homomorphism into $\mathbf{2}$.

Corollary

Apart of rules with contradictive premises there are no passive rules in any intermediate logic L. Hence, if L is actively structurally complete, then it is structurally complete.

Projective algebras

Unification

Let $\mathcal{Q}$ be a variety and $E(\bar{x})$ be a finite set of equations. An **unifier** of $E(\bar{x})$ is a substitution $s\colon \boldsymbol{Form}(\bar{x}) \to \boldsymbol{Form}$ such that $\mathcal{Q} \models \forall \bar{x} \bigwedge s(E(\bar{x}))$. This condition is equivalent to $\boldsymbol{Fr}_{\mathcal{Q}} \models s(E(\bar{x}))$ and, by what we already know, to the existence of a homomorphism $u\colon \boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/E \to \boldsymbol{Fr}_{\mathcal{Q}}$.

$$
\begin{tikzcd}
\boldsymbol{Form}(\bar{x}) \arrow[r, "s"] \arrow[d, "e"'] & \boldsymbol{Form} \arrow[d] \\
\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/E \arrow[r, dashed, "u"'] & \boldsymbol{Fr}_{\mathcal{Q}}
\end{tikzcd}
$$

By a **unifier** we also mean any homomorphism from a finitely presented algebra to a free algebra.
A finite set E of equations is **unifiable** if it admits a unifier.

Remark

Unification is relevant in computer science, in particular, in logic programming. It is a part of the resolution method.

Remark

One would define unifiers a quasivariety. Note however that a set of equations unifiable in $\mathcal{Q}$ would be unifiable iff it is unifiable in $V(\mathcal{Q})$, even if $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/E$ and $\boldsymbol{Fr}_{V(\mathcal{Q})}(\bar{x})/E$ differ.
Projectivity, will be defined in a moment, is also the same for $\mathcal{Q}$ and $V(\mathcal{Q})$.

Remark

We may define unifiers as homomorphisms into free algebras of finite or infinite rank. In practice, it is generally convenient to deal with free algebras of finite rank.

Remark

There exists a slightly different algebraic version of unifications. Ghilardi considers as unifiers homomorphisms from finitely presented algebras into finitely presented projective algebras. This approach has advantages, but a difference will not affect this lecture.

Definition

Let $\boldsymbol{P}$ be a finitely presented algebra and $u\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{V}}$ be its unifier. Then u is a **most general unifier** if for any other unifier $u\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{V}}$ there exists a homomorphism $g\colon \boldsymbol{Fr}_{\mathcal{V}} \to \boldsymbol{Fr}_{V}$ such that $v = g \circ u$.

$$\begin{tikzcd} & \boldsymbol{P} \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \boldsymbol{Fr}_{\mathcal{V}} \arrow[rr, dashed, "g"'] & & \boldsymbol{Fr}_{\mathcal{V}} \end{tikzcd}$$

$\mathcal{V}$ has **unary unification type** if every finitely presented algebra in $\mathcal{V}$ has a most general unifier.

Projective unification

Definition

An algebra $\boldsymbol{Pr}$ is **projective** in $\mathcal{V}$ if for any homomorphism $h \in \boldsymbol{A} \to \boldsymbol{B}$, $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$, and a surjective homomorphism $v\colon \boldsymbol{Pr} \to \boldsymbol{B}$ there exists a homomorphism $\bar{v}\colon \boldsymbol{Pr} \to \boldsymbol{A}$ such that $v = h \circ \bar{v}$.

$$\begin{tikzcd}
& \boldsymbol{Pr} \arrow[dl, dashed, "\bar{v}"'] \arrow[dr, "v"] & \\
\boldsymbol{A} \arrow[rr, two heads, "h"'] & & \boldsymbol{B}
\end{tikzcd}$$

Definition

An algebra $\boldsymbol{A}$ is a **retract** of an algebra $\boldsymbol{B}$ if there are homommorphsisms $i\colon \boldsymbol{A} \to \boldsymbol{B}$ and $r\colon B \to \boldsymbol{A}$ such that $r \circ i = id_A$

Exercise

An algebra (generated by X) in projective in $\mathcal{V}$ iff it is a retract of a free algebra (generated by X).

Fact

Let $\mathcal{V}$ be a variety and $\boldsymbol{P}$ be finitely presented in $\mathcal{V}$. If $\boldsymbol{P}$ is projective then it has a most general unifier.

Proof.
Let $i\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{V}}$ and $r\colon \boldsymbol{Fr}_{\mathcal{V}} \to \boldsymbol{P}$ be such that $r \circ i = id_P$. Then i is a most general unifier for $\boldsymbol{P}$. Indeed, suppose that $v\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{V}}$ is an unifier for $\boldsymbol{P}$. Let $g = v \circ r\colon \boldsymbol{Fr}_{\mathcal{V}} \to \boldsymbol{Fr}_{\mathcal{V}}$. Then
$v = v \circ id_P = v \circ r \circ i = g \circ i$.

Definition

A variety has projective unification if every its finitely presented unifiable algebra is projective.

Thus, a variety with projective unification has unitary unification type.

Fact (Dzik)

Let $\mathcal{Q}$ be a quasivariety with projective unification. Then $\mathcal{Q}$ is actively structurally complete.

Proof.

Let $\boldsymbol{P}$ be unifiable finitely presented algebra. By assumption $\boldsymbol{P}$ is projective. Hence it it s retract of a free algebra. In particylar, $\boldsymbol{P} \in S(\boldsymbol{Fr}_{\mathcal{Q}}) \subseteq Q(\boldsymbol{Fr}_{\mathcal{Q}})$.

One may also derive a fact concerning particular formulas.

Fact

Let $\mathcal{Q}$ be a quasivariety and $\boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/E$ be projective in $\mathcal{Q}$. Then, for every quasi-identity $q : \forall \bar{x} E \implies \varphi = \psi$, we have

$$q \text{ admissible in } \mathcal{Q} \text{ iff } q \text{ derivable in } \mathcal{Q}.$$

Proof.

$(\Rightarrow)$ Let $\boldsymbol{A} \in \mathcal{Q}$ and suppose that $\boldsymbol{A} \vDash E(\bar{a})$ for some $\bar{a}$. This means that there is a homomorphism $h\colon \boldsymbol{P} = \boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})/E \to \boldsymbol{A}$ such that $\boldsymbol{A} \vDash h(E)$. We have to show that $h(\varphi) = h(\psi)$. Recall that there is $i\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{Q}}(\bar{x})$ such that $i \circ r = id_P$, where $r(x) = x$. Thus, by admissibility,

$$h(\varphi) = h \circ r \circ i(\varphi) = h \circ r \circ i(\psi) = h(\psi).$$

Definition

A variety is **discriminator variety** if there are a class $\mathcal{S}$ of algebras and a so called **switching formula** sw such that $\mathcal{V} = V(\mathcal{S})$ and in every $S \in \mathcal{S}$ we have

$$sw(a, b, c, d) = \begin{cases} c & \text{if } a = b \\ d & \text{if } a \neq b \end{cases}$$

Theorem (Burris)

Every discriminator variety has projective unification.

Proof. It may be shown that s is a switching formula on all subdirectly irreducible algebras in $\mathcal{V}$ (which b.d.w. are simple). Let $\boldsymbol{P} = \boldsymbol{Fr}_{\mathcal{V}}(\bar{x})/(\varphi = \psi)$ (just a special case) be unifiable.
Let $u\colon \boldsymbol{P} \to \boldsymbol{Fr}(\bar{x})$ be any unifier. Define $i'\colon \boldsymbol{Fr}_{\mathcal{V}}(\bar{x}) \to \boldsymbol{Fr}_{\mathcal{V}}(\bar{x})$ by

$$i'(x_i) = sw(\varphi, \psi, x_i, u(x_i))$$

Proof cont.. We check that $i'(\varphi) = i'(\psi)$, and hence, that there exists $i\colon \boldsymbol{P} \to \boldsymbol{Fr}_{\mathcal{V}}(\bar{x})$ such that $i' = i \circ e$. Let $\boldsymbol{Fr}_{\mathcal{V}}(\bar{x}) \leqslant_{SD} \prod \boldsymbol{S}_k$ be a subdirect representation, where $\boldsymbol{S}_k$ are subdirectly irreducible. It is enough to show that $p \circ i'(\varphi) = p \circ i'(\psi)$, where p is any projection on $\boldsymbol{S}_k$.

$$
\begin{aligned}
p \circ i'(\varphi(\bar{x})) &= \varphi(p(sw(\varphi, \psi, x_1, u(x_1))), \dots) \\
&= \varphi(sw(p(\varphi), p(\psi), p(x_1), p(u(x_1))), \dots) \\
&= \begin{cases} \varphi(p(x_1), \dots) \\ \varphi(p(u(x_1)), \dots) \end{cases} = \begin{cases} p(\varphi(x_1, \dots)) & \text{if } p(\varphi) = p(\psi) \\ p(\varphi(u(x_1), \dots)) & \text{if } p(\varphi) \neq p(\psi) \end{cases} \\
&= \begin{cases} p(\psi(x_1, \dots)) & \text{if } p(\varphi) = p(\psi) \\ p(\psi(u(x_1), \dots)) & \text{if } p(\varphi) \neq p(\psi) \end{cases} = p \circ i'(\psi(\bar{x}))
\end{aligned}
$$

Proof cont.. We finally show that $r\colon \boldsymbol{Fr}(x_i) \to \boldsymbol{P}$, given by $r(x_i) = x_i$, is a retraction. More specifically, that $r \circ i = id_P$. It is enough to check it on generators. Since $\boldsymbol{P} \vDash \varphi = \psi$, we have

$$r \circ i(x_i) = sw(\varphi, \psi, x_i, u(x_i)) = x_i.$$

Remark

There is a skip in the last step. Namely, *sw* does not need to be a switching term in $\boldsymbol{P}$. However, $\mathcal{V} \vDash sw(a, a, c, d) = c$. Justify it with the use of subdirect representation.

There are plenty examples of discriminator varieties:

- The variety boolean algebras;
- The variety $V(\boldsymbol{L}_n)$ of MV-algebras.
- The variety of monadic alagebras $\mathcal{V}_{S5}$.
- The variety of cylindrical algebras of dimension n.
- The variety of rings satisfying $x^n = 0$.
- The variety of relation algebras.
- The variety of double Heyting algebras satisfying $(\sim\neg)^{n+1}x = (\sim\neg)^n x$.
- Almost every finite algebra has a switching term (Murskiĭ).

And many more. And all their subvarieties.

Theorem (Campercholi, S., Vaggione)

Let $\mathcal{V}$ be a discriminator variety in a finite (infinite) language. Then $\mathcal{V}$ is structurally complete iff $\mathcal{V}$ is minimal or (an elementary extension of) $\boldsymbol{Fr}_{\mathcal{V}}$ has an idempotent element.

Corollary

It is common for a logic L to have distinguished constants for verum and falsum. If L is algebraizable and its algebraic counterpart is a discriminator variety then L is structurally complete iff it has at most one proper axiomatic extension, and this extension is inconsistent.

Although there are plenty of examples of discriminator varieties. It is not difficult to find a non-discriminator one with projective unification. Indeed, every subdirectly irreducible algebra must be simple.

Exercise

Let L be an intermediate or a modal algebra. Suppose that there is a rooted frame $\mathbb{F}$ having a proper non-empty generated subframe.
If $\mathbb{F} \models \mathsf{L}$ then $\mathcal{V}_{\mathsf{L}}$ is not a discriminator variety.
In particular, if  $\models \mathsf{L}$ then $\mathcal{V}_{\mathsf{L}}$ is not a structurally complete.

Theorem (Wroński)

Let L be an intermediate logic. Then L has projective unification iff $LC \subseteq L$.

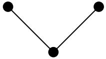
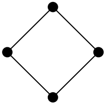
Definition

A frame $\mathbb{I}$ is injective in a class of $\mathcal{C}$ of frames iff For every injective p-morphism $j\colon \mathbb{F} \to \mathbb{G}$, $\mathbb{F}, \mathbb{G} \in \mathcal{C}$, and a p-morphism $f\colon \mathbb{F} \to \mathbb{I}$ there exists $\bar{f}\colon \mathbb{G} \to \mathbb{I}$ such that $f = \bar{f} \circ j$. Let L be an intermediate logic. Then L has projective unification iff $LC \subseteq L$.

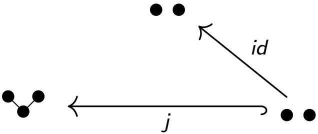
$$\begin{tikzcd} & \mathbb{I} & \\ \mathbb{G} \arrow[ur, dashed, "\bar{f}"] & & \mathbb{F} \arrow[ul, "f"'] \arrow[ll, hook', "j"] \end{tikzcd}$$

Proof.

Suppose that $LC \notin L$. Then $\mathbb{F}ork \vDash$ L or $\mathbb{D}iamond \vDash$ L.

$\mathbb{F}ork$   $\mathbb{D}iamond$

In the first case the frame $\bullet\ \bullet$ is not injective. Indeed, the diagram



cannot be completed. Hence $(\bullet\ \bullet)^*$ is not projective. In the second case we infer that ()* is not projective.

Proof cont..
Recall that $\mathcal{V}_{\mathrm{LC}}$, be the algebraic counterpart of LC, is locally finite. Thus, we may use finite duality. Recall that every finite frame $\mathbb{F}$ for L is a co-forest (all principal upsets are chains). Let $\mathbb{G}$ be a non-empty generated subframe of $\mathbb{F}$. Let w be any maximal element in $\mathbb{G}$. Define $e\colon \mathbb{G} \to \mathbb{F}$, $e(v) = v$, and $f\colon \mathbb{F} \to \mathbb{G}$ by

$$f(v) = \begin{cases} v & \text{if } v \in G \\ u & \text{if } v < u \text{ and } u \text{ is minimal in } \mathbb{G} \\ u & \text{otherwise} \end{cases}$$

Then $f \circ e = id_G$ and $e^* \circ f^* = id_{G^*}$. This shows that $\mathbb{G}^*$ is a retract of $\mathbb{F}^*$.
We conclude that every finite nontrivial algebra in $\mathcal{V}_{\mathrm{LC}}$ is projective.

A **Hilbert** algebra is an algebra $\boldsymbol{A} = (A, \to, 1)$ is which an identity is valid if it is valid in all Heyting algebras. Let $\mathcal{HIA}$ be the variety of Hilbert algebras.

Coroolary (Prucnal)

Let $\boldsymbol{P}$ be a nontrivial finitely presneted Hilber algebra. Then $\boldsymbol{P}$ is projective. Hence, $\mathcal{HIA}$ is (hereditary) structurally complete.

Proof.

Let $\boldsymbol{P} = \boldsymbol{Fr}(\bar{x})/\varphi = \boldsymbol{Fr}(\bar{x})/\alpha$. Let $\boldsymbol{P} \xrightarrow{i} \boldsymbol{Fr}(\bar{x}) \xrightarrow{r} \boldsymbol{P}$ be given by

$$i(a) = \varphi \to a, \qquad r(\varphi) = \psi/\alpha$$

- i is well defined since $i(\varphi) = \varphi \to \varphi = 1$;
- i is a homomorphism since $\mathcal{HIA}, \mathcal{HA} \models \forall x \; x \to 1 = 1$ and $\mathcal{HIA}, \mathcal{HA} \models \forall x, y, z \;\; x \to (y \to z) = (x \to y) \to (x \to z)$;
- and $r \circ i = id_P$ since in $\boldsymbol{P}$, $\varphi \to a = 1 \to a = 1$.

Fact

Essentially the same argument shows (hereditary) structural completeness for $\{\to\}$-, $\{\to, \wedge\}$-, $\{1, \to, \wedge\}$-fragments of INT.

Theorem (Cintula, Metcalfe)

But $\{\to, \vee\}$- and $\{\to, 0\}$-fragments of INT are not structurally complete.

There are other results of this sort.

Theorem (Dzik, Wojtylak)

Let L be a normal extension of S4 modal logic. Then L has projective unification iff $\mathsf{S4.3} \subseteq \mathsf{L}$.

S4.3 is the logic of chains of clusters

Theorem (Kost)

Let L be a normal extension of K4 modal logic. Then L has projective unification iff $\mathsf{K4.3} \oplus \Box(\Box x \to x) \subseteq \mathsf{L}$.

$\mathsf{K4.3} \oplus \Box(\Box x \to x)$ is the logic of chain of clusters plus, possibly, one irreflexive vertex at the bottom.

Exercise

Show that $\circ$ is not projective in the class containing  .

The proofs Dzik's, Wojtylak's and Kost's theorems are more demanding then the proof of Wroński's theorem for intermediate logics. The difficulty is in the fact that the corresponding varieties are not locally finite. Thus finite duality is not enough. Kost used Ghilardi's characterization of projectivity.

Definition (Ghilardi)

Let L be a normal extension of S4 modal logic (for simplicity) with the finite model property. A formula $\varphi(\bar{x})$ is **Ghilardi projective** in L if for every finite rooted frame $\mathbb{F} \models L$, with a root r and its cluster $\mathbb{C}_r$, every valuation val$'$ of $\mathbb{F} \setminus \mathbb{C}_r$ such that $(\mathbb{F} \setminus \mathbb{C}_r, val') \models \varphi$ there exists a valuation val of $\mathbb{F}$ extending val$'$ (i.e., $\mathrm{val}'(x) = \mathrm{val}(x) \setminus \mathbb{C}_r$) such that $(\mathbb{F}, \mathrm{val}) \models \varphi$.

Ghilardi projectivity

Definition

Let L be a normal extension of S4 modal logic (for simplicity) with the finite model property. A formula $\varphi(\bar{x})$ is **Ghilardi projective** in L if for every finite rooted frame $\mathbb{F} \models L$, with a root r and its cluster $\mathbb{C}_r$, every valuation val$'$ of $\mathbb{F} \setminus \mathbb{C}_r$ such that $(\mathbb{F} \setminus \mathbb{C}_r, val') \models \varphi$ there exists a valuation val of $\mathbb{F}$ extending val$'$ (i.e., $\mathrm{val}'(x) = \mathrm{val}(x) \setminus \mathbb{C}_r$) such that $(\mathbb{F}, \mathrm{val}) \models \varphi$.

Definition

Let $\mathcal{V}$ be a variety and $\boldsymbol{P}$ be finitely presented in $\mathcal{V}$ then $\boldsymbol{P}$ is **Ghilardi projective** if for every subdirectly irreducible algebra $\boldsymbol{S} \in V$, its monolith μ, and any homomoprhism $h\colon \boldsymbol{P} \to \boldsymbol{S}/\mu$ there exists a homomorphism $\bar{h}$ such that $h = \bar{h} \circ \mathrm{nat}\,\mu$.

$$\begin{tikzcd}
 & \boldsymbol{S} \arrow[dd, two heads, "\mathrm{nat}\,\mu"] \\
\boldsymbol{P} \arrow[ur, dashed, "\bar{h}"] \arrow[dr, "h"'] & \\
 & \boldsymbol{S}/\mu
\end{tikzcd}$$

Exercise

Write a definition of Ghilardi projectivity for intermediate logics.

Exercise

Convince yourself that the both definitions for Ghilardi projectivity mean the same.

$$\begin{tikzcd}
& \boldsymbol{S} \arrow[dd, two heads, "\mathrm{nat}\,\mu"] \\
\boldsymbol{P} \arrow[ur, dashed, "\bar{h}"] \arrow[dr, "h"'] & \\
& \boldsymbol{S}/\mu
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{F} \arrow[dl, dashed, "\bar{f}"'] \\
\mathbb{I} & \\
& \mathbb{F} \setminus \mathbb{C}_r \arrow[ul, "f"] \arrow[uu, hook, "j"']
\end{tikzcd}$$

Exercise (Prucnal)

Show that every formula $\neg\varphi$ is Ghilardi projective in any intermediate logic.

Exercise (Ghilardi)

Show that every formulas $x \to \varphi$ and $\varphi \to x$ are Ghilardi projective in any intermediate logic.

Theorem (Ghilardi)

Let L be intermediate logic or a normal extension of K4 modal logic with the finite model property. Then a formula is projective in L iff it is Ghilardi projective in L.

The proof is based on a skillful mixing two Prucnal's substitutions.

Dzik's and Wojtylak's theorem

Proof in non-easy direction.
Let $\mathbb{I}$ be a dual to a finitely presented unifiable S4.3-algebra. Recall that a finite rooted frame $\mathbb{F}$ for S4.3 logic is chains of clusters. Let $\mathbb{C}_r$ the the root's cluster in $\mathbb{F}$. Suppose that we have a p-morphism $f\colon \mathbb{F} \setminus C_r \to \mathbb{I}$. Our aim is to show that there is an extension $\bar{f}\colon \mathbb{F} \to \mathbb{I}$ of f.

Exercise

Use the fact that $\mathbb{I}^*$ is unifiable to show the existence of an extension in the case when $\mathbb{F} = \mathbb{C}_r$.

Suppose that $\mathbb{F} \neq \mathbb{C}_r$ and chose s from the lowest cluster in $\mathbb{F} \setminus \mathbb{C}_r$. Let $g\colon \mathbb{F} \to \mathbb{F} \setminus \mathbb{C}_r$ be the p-morphism given by

$$g(v) = \begin{cases} v & \text{if } v \notin \mathbb{C}_r \\ s & \text{if } v \in \mathbb{C}_r \end{cases}$$

Let $\bar{f} = f \circ g$. Then

$$\begin{aligned} \bar{f} \circ j &= f \circ g \circ j \\ &= f \circ id_{F \setminus C_r} = f. \end{aligned}$$

Wait. A finitely presented S4.3-algebra may be infinite. So why we consider just a frame $\mathbb{I}$?

Yes, this is a problem. But not a small one. $\mathbb{I}$ may be infinite, but all downsets in $\downarrow^{\mathbb{I}} v$ are closed. thus g is continuous.

Opinion

A great feature of Ghilardi projectivity is that it allows to keep considerations on finite (topology-less) level.

Interesting fact

Let $\boldsymbol{S}$ be a finite subdirectly irreducible algebra and $\mathcal{V} = V(\boldsymbol{S})$. Then $\boldsymbol{S}$ embeds into $\boldsymbol{Fr}_{\mathcal{Q}}$ iff it is projective in $\mathcal{V}$.

Proof. Let $i\colon \boldsymbol{S} \to \boldsymbol{Fr}_{\mathcal{Q}}$ be injective. Recall that $\boldsymbol{Fr}_{\mathcal{Q}}$ is a subalgebra of $\boldsymbol{S}^K$ for some set K. Since $\boldsymbol{S}$ subdirrectly irreducible, there exists $k \in K$ such that $p_k \circ i\colon \boldsymbol{S} \to \boldsymbol{S}$ is an embedding. Since $\boldsymbol{S}$ is finite, $p_k \circ i$ is an automorphism. Again by finiteness, there is n such that $(p_k \circ i)^n = id_S$. Then $p_k \circ (i \circ p_k)^{n-1}$ is a retraction.

Theorem

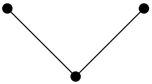
Let $\boldsymbol{A}$ be a finite algebra and $\mathcal{V} = V(\boldsymbol{A})$. Then $\mathcal{V}$ is structurally complete iff the conditions hold:

- Every subdirectly irreducible algebra in $\mathcal{V}$ has cardinality at most $|A|$;
- Every subdirectly ireducible algebra in $\mathcal{V}$, that does not embed properly in any other, is projective in $\mathcal{V}$.

Exercise

Show that the logic of $\mathbb{F}ork$ is actively structurally complete , but is does not have projective unification.

$\mathbb{F}ork$



Project proposal

Is there any variant of the above theorem in the infinite setting? In particular, when (finitely presented) subalgebas of free algebras are projective?

Remark

Interestingly, every finitely presented subalgebras of a free Heyting algebras is projective. But it is a non-trivial fact observed by Ghilardi.